Inequalities

Daniel Spivak
d2spivak@uwaterloo.ca

Rearrangement Inequality and Chebychev's Inequality

If $x_1 \leq x_2 \leq ... \leq x_n, y_1 \leq y_2 \leq ... \leq y_n, z_1, z_2, ..., z_n$ is a permutation of $y_1, y_2, ..., y_n$, then
$\sum\limits_{i=1}^{n} x_i y_i \geq \sum\limits_{i=1}^{n} x_i z_i$.
Similarly, if $x_1 \leq x_2 \leq ... \leq x_n, y_1 \geq y_2 \geq ... \geq y_n, z_1, z_2, ..., z_n$ is a permutation of $y_1, y_2, ..., y_n$, then
$\sum\limits_{i=1}^{n} x_i y_i \leq \sum\limits_{i=1}^{n} x_i z_i$.
Summing over all possible permutations of $y_1, y_2, ..., y_n$, we get Chebyshev's Inequality:
If $x_1 \leq x_2 \leq ... \leq x_n, y_1 \leq y_2 \leq ... \leq y_n$, then
$n \sum\limits_{i=1}^{n} x_i y_i \geq \sum\limits_{i=1}^{n} x_i \sum\limits_{i=1}^{n} y_i$.
Similarly, if $x_1 \leq x_2 \leq ... \leq x_n, y_1 \geq y_2 \geq ... \geq y_n$, then
$n \sum\limits_{i=1}^{n} x_i y_i \leq \sum\limits_{i=1}^{n} x_i \sum\limits_{i=1}^{n} y_i$.

AM-GM and similar inequalities

If $a_1, a_2, ..., a_n \geq 0$, then
$\frac{a_1+a_2+...+a_n}{n} \geq \sqrt[n]{a_1 a_2 ... a_n}$.
This can be generalized to the **RMS-AM-GM-HM**, which states that:
If $a_1, a_2, ..., a_n \geq 0$, then
$\sqrt{\frac{a_1^2+a_2^2+...+a_n^2}{2}} \geq \frac{a_1+a_2+...+a_n}{n} \geq \sqrt[n]{a_1 a_2 ... a_n} \geq \frac{n}{\frac{1}{a_1}+\frac{1}{a_2}+...+\frac{1}{a_n}}$.
Even more generally, the **Power Mean Inequality** states that:
If $a_1, a_2, ..., a_n \geq 0$ and $w > x > 0 > y > z$, then
$(a_1^w+a_2^w+...+a_n^w)^{\frac{1}{w}} \geq (a_1^x+a_2^x+...+a_n^x)^{\frac{1}{x}} \geq \sqrt[n]{a_1 a_2 ... a_n} \geq (a_1^y+a_2^y+...+a_n^y)^{\frac{1}{y}} \geq (a_1^z + a_2^z + ... + a_n^z)^{\frac{1}{z}}$.
These have equality if and only if $a_1 = a_2 = ... = a_n$.
AM-GM can also be used to prove the **Cauchy-Schwarz** inequality, which states that:
For any sequences of real numbers $x_1, x_2, ..., x_n$ and $y_1, y_2, ..., y_n$,
$(\sum\limits_{i=1}^{n} x_i y_i)^2 \leq \sum\limits_{i=1}^{n} x_i^2 \sum\limits_{i=1}^{n} y_i^2$.
Cauchy-Schwarz is a special case of **Holder's inequality**, in which $p = q = 2$.
In general:

If p and q are positive real numbers such that $\frac{1}{p}+\frac{1}{q}=1$, then for all sequences of non-negative real numbers $x_1, x_2, ..., x_n$ and $y_1, y_2, ..., y_n$,
$\sum\limits_{i=1}^{n} x_i y_i \leq (\sum\limits_{i=1}^{n} x_i^p)^{\frac{1}{p}} (\sum\limits_{i=1}^{n} y_i^q)^{\frac{1}{q}}$.
In Holder's inequality, the equality case occurs when $\frac{x_1^p}{y_1^q} = \frac{x_2^p}{y_2^q} = ... = \frac{x_n^p}{y_n^q}$
Note: Holder's inequality is spelled with a fancy, double-dotted o, but it does not show up in this editor.

Miscellaneous useful theorems

Jensen's Inequality allows is one of the few ways in which you can use derivatives without making graders get angry:
Suppose $f(x)$ is a convex function on $[a, b]$, that is to say, $f''(x) \geq 0$ if $x \in [a, b]$. Then for all positive real numbers $t_1, t_2, ..., t_n$ and all $x_1, x_2, ..., x_n \in [a, b]$ such that $t_1 + t_2 + ... + t_n = 1$, we have
$t_1 f(x_1) + t_2 f(x_2) + ... + t_n f(x_n) \geq f(t_1 x_1 + t_2 x_2 + ... + t_n x_n)$,
and if $f''(x) > 0$, then this is with equality if and only if $x_1 = x_2 = ... = x_n$.
Schur's Inequality is the first one to suspect when there is a weird equality case - equality holds when two variables are equal and the third is also equal or is zero:
Let $f(x)$ be a non-decreasing, non-negative function, e.g. $f(x) = x^k$ with $k \geq 0$. Then for all non-negative real a, b, c,
$f(a)(a-b)(a-c) + f(b)(b-a)(b-c) + f(c)(c-a)(c-b) \geq 0$.
The special case of $f(x) = x$ yields
$a^3 + b^3 + c^3 + 3abc \geq a^2 b + ab^2 + a^2 c + ac^2 + b^2 c + bc^2$.
A rarely-used but very **Fancy version of Schur's** has five variables, but without the flexibility of a function:
If a, b, c, d, e are any real numbers, then
$(a-b)(a-c)(a-d)(a-e) + (b-a)(b-c)(b-d)(b-e) + (c-a)(c-b)(c-d)(c-e) + (d-a)(d-b)(d-c)(d-e) + (e-a)(e-b)(e-c)(e-d) \geq 0$.
Equality occurs whenever each variable is equal to at least one other variable.
Muirhead's Inequality is very helpful when bashing, although it frequently does not apply until you apply AM-GM several times:
A sequence $a_1, a_2, ..., a_n$ majorizes a sequence $b_1, b_2, ..., b_n$ when $x_1 \geq x_2 \geq ... \geq x_n, y_1 \geq y_2 \geq ... \geq y_n, x_1 + x_2 + ... + x_n = y_1 + y_2 + ... + y_n$ and
$a_1 + a_2 + ... + a_k \geq b_1 + b_2 + ... + b_k$ for $k = 1, 2, ..., n$. In this case, for any sequence of non-negative real numbers $x_1, x_2, ..., x_n$,
$\sum\limits_{sym} x_1^{a_1} x_2^{a_2} ... x_n^{a_n} \geq \sum\limits_{sym} x_1^{b_1} x_2^{b_2} ... x_n^{b_n}$.
Note that the sum is symmetric, not cyclic, and that the theorem is often false

in the cyclic case.
And for all those pesky inequalities with exponents, **Bernoulli's Inequality** states that:
For all $k \geq 1$ and $x \geq -1$,, $(1+x)^r \geq 1 + xr$

Common techniques

An inequality is homogenous if when all variables are multiplied by a constant, the inequality is unaffected, i.e. the total exponent of all variables in each term is the same. If you are trying to bash and have a non-homogenous condition, you can multiply by the condition to homogenize and make bashing easier. For example, if $xyz = 8$ is given, and the inequality to be homogenized is $x^2 + y \leq 5$, it can be changed to $x^2 + y\frac{(xyz)^{\frac{1}{3}}}{2} \leq 5\frac{(xyz)^{\frac{2}{3}}}{4}$.
Also, if you are given that the product of some variables is 1, you can force homogeneity by setting each variable as a quotient; for example, if $abc = 1$, you may set $a = \frac{x}{y}, b = \frac{y}{z}, c = \frac{z}{x}$.
If you are given the silly condition that a, b, and c are sides of a triangle, introduce new positive variables x, y and z such that $a = x + y$, $b = x + z$, and $c = y + z$.
You should always try to guess the equality cases of the inequality; if an equality case involves one or more of the variables being zero, it's likely that Schur's inequality is involved. This will also help you determine what weights to use when you apply weighted AM-GM.
Expanding power series: This is most frequently seen with $\frac{1}{1-x} = 1 + x + x^2 + x^3 + ...$ for $-1 < x < 1$. While the original fraction is difficult to apply AM-GM on, power series are just meant to have roots taken of them.
Factoring: If you have a polynomial in two variables, you can factor it to find out precisely when it is bigger or smaller than zero; this is particularly useful if you've reduced the inequality to one variable by
Showing that variables are equal: If you manage to show that replacing a, b with $\frac{a+b}{2}, \frac{a+b}{2}$ or any other measure of central tendency preserves the inequality, most frequently by using Jensen's inequality, then you can make many variables equal and reduce the number of variables enough to do this.

Problems

To build your confidence:

1. Let x, y, z be positive real numbers. Show that
$x^2 + y^2 + z^2 + xy + xz + yz \geq 6$.

2. Let x, y, z be positive real numbers. Show that
$\frac{x}{y+z} + \frac{y}{x+z} + \frac{z}{x+y} \geq \frac{3}{2}$.
3. Let a, b, x, y be positive real numbers. Show that
$(a^4 + b^4)(x^4 + y^4) + (a^3b + ab^3)(x^3y + xy^3) + 4abxy \geq$
$\frac{1}{3}(a^4 + a^3b + 2a^2b^2 + ab^3 + b^4)(x^4 + x^3y + 2x^2y^2 + xy^3 + y^4)$.
4. (CMO 2012) Let x, y, z be positive real numbers. Show that
$x^2 + xy^2 + xyz^2 \geq 4xyz - 4$.
5. Prove that for any two sequences of positive real numbers $a_1, a_2, ..., a_n$ and $b_1, b_2, ..., b_n$,
$\frac{a_1^2}{b_1} + \frac{a_2^2}{b_2} + ... + \frac{a_n^2}{b_n} \geq \frac{(a_1+a_2+...+a_n)^2}{b_1+b_2+...+b_n}$.

Not too difficult:

6. Let x, y, z be positive real numbers. Show that
$(xyz)^2 + x^2 + y^2 + z^2 + 2 \geq 2(xy + xz + yz)$.
7. Let a, b, c, d be positive real numbers. Show that
$a^4b + b^4c + c^4d + d^4a \geq abcd(a + b + c + d)$.
8. Let a, b, c be positive real numbers with $abc = 1$. Show that
$a^3 + b^3 + c^3 + 6 \geq 3ab + 3bc + 3ac$.
9. Let a, b, c be positive real numbers with $a + b + c = 1$. Show that
$a^3 + b^3 + c^3 + 6abc \geq \frac{1}{4}$.
10. (IMO 1974) If a, b, c, d are positive reals, determine the possible values of
$\frac{a}{a+b+d} + \frac{b}{a+b+c} + \frac{c}{b+c+d} + \frac{d}{a+c+d}$

Challenging:

11. (CMO 1999) Let x, y, z be non-negative real numbers with $x + y + z = 1$. Show that $x^2y + y^2z + z^2x \leq \frac{4}{27}$, and determine when equality occurs.
12. (USAMO 2004) Let a, b, c be positive real numbers. Show that
$(a^5 - a^2 + 3)(b^5 - b^2 + 3)(c^5 - c^2 + 3) \geq (a + b + c)^3$
13. (IMO 2004) Let $n \geq 3$ be an integer. Let $t_1, t_2, ..., t_n$ be positive real numbers such that $n^2 + 1 > (t_1 + t_2 + ... + t_n)(\frac{1}{t_1} + \frac{1}{t_2} + ... + \frac{1}{t_n})$.
Show that for all i, j, k with $1 \leq i < j < k \leq n$, t_i, t_j and t_k are the sides of a triangle.
14. (Chinese TST 2003) Let x, y, z be positive real numbers with $x+y+z = xyz$. Find the minimum value of $x^7(yz - 1) + y^7(zx - 1) + z^7(xy - 1)$
15. (USAMO 1998) Let $a_0, a_1, ..., a_n$ be real numbers in the interval $(0, \frac{\pi}{2})$ such that $\tan(a_0 - \frac{\pi}{4}) + \tan(a_1 - \frac{\pi}{4}) + ... + \tan(a_n - \frac{\pi}{4}) \geq n - 1$. Prove that
$\tan(a_0)\tan(a_1)...\tan(a_n) \geq n^{n+1}$.

To crush your hopes:

16. (ISL 1998) Let x, y and z be positive real numbers such that $xyz = 1$. Prove that $\frac{x^3}{(1+y)(1+z)} + \frac{y^3}{(1+z)(1+x)} + \frac{z^3}{(1+x)(1+y)} \geq \frac{3}{4}$.
17. (USAMO 2001) Let $a, b, c \geq 0$ satisfy $a^2 + b^2 + c^2 + abc = 4$. Show that

$0 \le ab + bc + ca - abc \le 2$.
18. (Chinese TST 2005) Let a, b, c be non-negative integers such that $ab + ac + bc = \frac{1}{3}$. Prove that $\frac{1}{a^2-bc+1} + \frac{1}{b^2-ac+1} + \frac{1}{c^2-ab+1} \le 3$.
19. (Chinese TST 2009) Let nonnegative real numbers a_1, a_2, a_3, a_4 satisfy $a_1 + a_2 + a_3 + a_4 = 1$. Prove that
$\max\{\sum_{i=1}^{4} \sqrt{a_i^2 + a_i a_{i-1} + a_{i-1}^2 + a_{i-1}a_{i-2}}, \sum_{i=1}^{4} \sqrt{a_i^2 + a_i a_{i+1} + a_{i+1}^2 + a_{i+1}a_{i+2}}\} \ge 2$, where for all integers $i, a_{i+4} = a_i$ holds.
20. (ISL 2003) Let n be a positive integer and let $(x_1, x_2, ..., x_n), (y_1, y_2, ..., y_n)$ be two sequences of positive real numbers. Suppose $(z_2, z_3 ..., z_{2n})$ is a sequence of positive real numbers such that $z_{i+j}^2 \ge x_i y_j$ for all $1 \le i, j \le n$. Let $M = \max\{z_2, z_3, ..., z_{2n}\}$. Prove that
$\left(\frac{M+z_2+z_3+...+z_{2n}}{2n}\right)^2 \ge \left(\frac{x_1+x_2+...+x_n}{n}\right)\left(\frac{y_1+y_2+...+y_n}{n}\right)$